

Finitary Representations and Images of Transitive Finitary Permutation Groups

Felix Leinen

metadata, citation and similar papers at core.ac.uk

E-mail: leinen@mathematik.uni-mainz.de

and

Orazio Puglisi

*Dipartimento di Matematica, Università degli Studi di Trento,
I-38050 Povo (Trento), Italy*

E-mail: puglisi@alpha.science.unitn.it

Communicated by Peter M. Neumann

Received November 5, 1998

DEDICATED TO MARIO CURZIO ON THE OCCASION OF
HIS 70TH BIRTHDAY

We characterize the point stabilizers and kernels of finitary permutation representations of infinite transitive groups of finitary permutations. Moreover, the number of such representations is determined. © 1999 Academic Press

Key Words: finitary groups; permutation groups; wreath products; homomorphic images.

1. INTRODUCTION

In this note we are concerned with the question, which homomorphic images of an infinite transitive group of finitary permutations do again have faithful representations by finitary permutations? Recall that every *finitary permutation* of a set Ω fixes a cofinite subset of Ω . In particular, finitary permutation groups are locally finite. Finitary permutations are special in-

stances of *finitary transformations* of a vector space V , that is, transformations which fix a subspace of finite codimension in V . Groups of finitary transformations—so-called *finitary linear groups*—generalize the notion of linear groups.

It is well known that homomorphic images of linear groups need not be linear again (see [16, Sect. 6]). Even worse, B. A. F. Wehrfritz has given examples of linear groups with images, which do not admit any faithful finitary linear representation [17]. On the positive side, it is well known that Zariski closed normal subgroups of linear groups lead to linear factor groups (see [16, Theorem 6.4]). Moreover, there is the very remarkable observation, due to R. E. Phillips and J. G. Rainbolt, that every homomorphic image of a periodic linear group with trivial unipotent radical is linear [13]. This latter result has been generalized by B. Wehrfritz to quotients of arbitrary linear groups modulo periodic normal subgroups with trivial unipotent radical [18].

In view of these results it seems natural to ask which images of finitary linear groups are again finitary linear. The canonical extension of the Zariski topology does not behave satisfactorily in this respect [19]. And up to now there is just a collection of special instances of normal subgroups in finitary linear groups, which lead to finitary quotients [12, Theorem 8.5, 19]. Our aim here is therefore to give the best possible description of the situation in the very first case, when the group under consideration is a transitive group of finitary permutations. This should be quite instructive for the more general case of irreducible finitary linear groups, since every such imprimitive group acts as a transitive finitary permutation group on the set of finite-dimensional blocks of any system of imprimitivity [11, 2.2.3].

A full description of the structure of infinite transitive finitary permutation groups can be found in [9, 10, 14]. The only such primitive groups are the alternating group $\text{Alt}(\Omega)$ and the full finitary symmetric group $\text{FSym}(\Omega)$ (see [21, Satz 9.4]). The alternating group is simple and has index two in $\text{FSym}(\Omega)$. Both groups just admit their natural representation as the only faithful transitive finitary permutation representation (Proposition 2.4). Infinite imprimitive finitary permutation groups fall into two classes, *almost-primitive* groups and *totally imprimitive* groups. Almost-primitive groups contain just finitely many normal subgroups [14], and the results of [14] allow a straightforward description of their finitary images (see Section 2).

Therefore we are mainly concerned here with totally imprimitive subgroups of $\text{FSym}(\Omega)$. In this situation Ω is the union of an ascending chain of finite G -blocks, whence Ω and G are countable. The point stabilizers of totally imprimitive finitary permutation representations of G will be characterized as follows.

THEOREM A. *Let G be a totally imprimitive subgroup of $\text{FSym}(\Omega)$. The subgroup K of G is the point stabilizer of some totally imprimitive finitary permutation representation of G if and only if*

(A.1) *K is a subgroup of finite index in the setwise stabilizer $G_{\{\Delta\}}$ for some finite G -block Δ in Ω , and*

(A.2) *for every finite subgroup F of G , all but finitely many G -conjugates of F are contained in K .*

Note that every transitive finitary permutation representation of a totally imprimitive (resp. almost-primitive) group on an infinite set is totally imprimitive (resp. almost-primitive or primitive) again, since totally imprimitive groups are precisely those infinite transitive finitary permutation groups which do not contain an infinite simple section [14]. On the other hand, every subgroup of finite index in a transitive finitary permutation group G contains the derived subgroup G' , so that G' acts trivially in every permutation representation of G on a finite set [10, Theorem 1 and Lemma 2.1].

In the situation of Theorem A it will also turn out that K contains the derived subgroup of the pointwise stabilizer G_{Δ} for some finite G -block Δ in Ω (Proposition 3.4). This will allow us to derive the following criterion concerning the number of finitary permutation representations of G .

THEOREM B. *The totally imprimitive subgroup G of $\text{FSym}(\Omega)$ has uncountably many inequivalent totally imprimitive finitary permutation representations if and only if*

(B.1) *the point stabilizer H in G has an infinite elementary-abelian quotient H/N , and*

(B.2) *for every finite subgroup F of G , all but finitely many G -conjugates of F are contained in N .*

In this case G has 2^{\aleph_0} inequivalent faithful totally imprimitive finitary permutation representations.

Note that every totally imprimitive finitary permutation group has infinitely many inequivalent faithful totally imprimitive finitary permutation representations (Lemma 3.1), while an almost-primitive finitary permutation group just admits finitely many inequivalent almost-primitive finitary permutation representations (Theorem 2.5).

The conditions in Theorem B are particularly easy to use in the case where G is an iterated wreath product of infinitely many finite transitive permutation groups (Theorem 4.1), but we shall also derive a handy sufficient condition under which certain non(locally solvable) totally imprimitive finitary groups have uncountably many totally imprimitive finitary permutation representations (Theorem 4.2).

With the aid of Theorem A, kernels of finitary permutation representations of G can be described.

THEOREM C. *Let M be a proper normal subgroup of the totally imprimitive subgroup G of $\text{FSym}(\Omega)$.*

(a) *If M is the kernel of some finitary permutation representation with all orbits infinite, then for every finite subgroup F of G there exists a finite G -block Δ in Ω such that $\text{supp}_\Omega F \subseteq \Delta$ and $\langle F^G \rangle \cap M = \bigcap_{g \in G} (\langle F^G \rangle \cap M) G_{\Delta g}$.*

(b) *If for every finite subgroup F of G there exists a finite G -block Δ in Ω such that $\text{supp}_\Omega F \subseteq \Delta$ and $\langle F^G \rangle \cap M = \langle F^G \rangle \cap \bigcap_{g \in G} M G_{\Delta g}$, then M is the kernel of some finitary permutation representation with all orbits infinite.*

The right side of the equation in (a) consists of precisely those elements in $\langle F^G \rangle$ which act on each G -translate of Δ like some element from $\langle F^G \rangle \cap M$, while the right side of the equation in (b) consists of precisely those elements in $\langle F^G \rangle$ which act on each G -translate of Δ like some element from M . In both situations, $\langle F^G \rangle \cap M$ is a full direct product of finite groups with supports Δg ($g \in G$). The condition in (b) implies the condition in (a), and it is not clear whether the converse holds for kernels M of finitary permutation representations. At least we can show that the converse holds when G/M has no proper abelian normal subgroup (Theorem 6.2), or when M is contained in the normal closure of a finite subgroup of G (Theorem 6.3). The latter applies in particular in the case where G is an iterated wreath product of finite transitive permutation groups.

If the totally imprimitive group G is perfect, then the kernel of any finitary permutation representation of G satisfies the condition in part (a) of Theorem C. Example 6.4 will show, however, that not every normal subgroup of a perfect totally imprimitive finitary permutation group satisfies this condition.

We shall also consider finitary linear quotients of transitive finitary permutation groups. Every such quotient with trivial unipotent radical is a finitary permutation group again (Theorems 2.2, 2.3, 6.5).

Before embarking on proofs, let us recall a couple of basic properties of any infinite transitive subgroup G of $\text{FSym}(\Omega)$, which will be used frequently in the sequel and without further reference.

Since G contains just elements of finite support, any proper G -block in Ω is finite. In particular, every intransitive normal subgroup of G has finite orbits and hence finite exponent. The derived subgroup G' is the unique minimal transitive normal subgroup of G (see [10, Theorem 1]). In particular, G' is perfect, and every subnormal transitive subgroup of G is normal in G . Moreover, every normal subgroup of G' is normal in G (see [10, Theorem 3.3]), and G has a trivial center [20].

2. ALMOST-PRIMITIVE GROUPS

In this section images of almost-primitive subgroups G of $\text{FSym}(\Omega)$ will be considered. Every such group G satisfies $W' = G' \leq G \leq W$, where W denotes a wreath product of the form $F \text{ wr}_{\Sigma} \text{FSym}(\Sigma)$ (F finite) in its natural finitary action on $\Omega = F \times \Sigma$ (see [10, Theorem 3]). In particular, since W/W' is finite, every transitive normal subgroup of G has finite index in W . We shall therefore concentrate on intransitive normal subgroups M of G . Note also that every normal subgroup of G is normalized by W' and hence is normal in W .

In the sequel, B will denote the base group of W . Since the orbits of the intransitive normal subgroup M are finite, and since W/B contains an infinite simple section, M is contained in B . Let \tilde{M} be the full direct product, which is generated by the images of the projections of M into the components of B . Obviously $W/\tilde{M} \cong (F/F \cap \tilde{M}) \text{ wr}_{\Sigma} \text{FSym}(\Sigma)$, and $G\tilde{M}/\tilde{M}$ is a transitive subgroup of W/\tilde{M} , since it contains the derived subgroup of W/\tilde{M} .

PROPOSITION 2.1. *For every intransitive normal subgroup M of the almost-primitive finitary permutation group G the section $G \cap \tilde{M}/M$ is finite and coincides with the center of G/M .*

Proof. We adopt the notation introduced above. In addition, F_{σ} will denote the σ -component in B ($\sigma \in \Sigma$). The section \tilde{M}/M is finite by [14, Lemma 2(iv)]. The top group of W acts highly transitively on Σ . Hence for every $\tilde{m} \in \tilde{M} \cap F_{\sigma}$ and for every $w \in W$ there exists $m \in M$ such that m acts like \tilde{m} on $F \times \{\sigma\}$, and such that the intersection of the supports of w and m is contained in $F \times \{\sigma\}$. It follows that $[\tilde{m}, w] = [m, w] \in M$. Since $\tilde{M} = \text{Dr}_{\sigma \in \Sigma}(\tilde{M} \cap F_{\sigma})$, this shows that \tilde{M}/M is contained in the center of W/M . On the other hand W/\tilde{M} is an almost-primitive or primitive finitary permutation group, whence the transitive subgroup $G/G \cap \tilde{M} \cong G\tilde{M}/\tilde{M}$ of W/\tilde{M} must have a trivial center. ■

THEOREM 2.2. *Let M be an intransitive normal subgroup of the almost-primitive finitary permutation group G .*

- (a) *M is the kernel of a transitive finitary permutation representation of G if and only if G/M has a trivial center.*
- (b) *M is the kernel of a finitary permutation representation of G if and only if $G'/G' \cap M$ has a trivial center.*

Proof. (a) If G/M has a trivial center, then $M = G \cap \tilde{M}$ by Proposition 2.1. And the quotient $G/G \cap \tilde{M}$ is a transitive finitary permutation group. Conversely, if G/M is a transitive finitary permutation group, then its center must be trivial.

(b) From Proposition 2.1, the quotient $G'/G' \cap M$ has a trivial center if and only if $G' \cap M = G' \cap \tilde{M}$. Suppose first that $G' \cap M = G' \cap \tilde{M}$. Let $C = G'M$. Then $C \cap \tilde{M} = (G' \cap \tilde{M})M = M$. The quotient G/C is finite, and the quotient $G\tilde{M}/\tilde{M}$ is a finitary permutation group. Hence G/M is a finitary permutation group as subdirect product of these quotients.

Conversely, suppose that G/M is a finitary permutation group. Since $G \cap \tilde{M}/M$ is a central section of G/M , it must act trivially on every infinite (G/M) -orbit. Moreover, G' must act trivially on every finite G -orbit. Hence M contains $\tilde{M} \cap G'$, and $\tilde{M} \cap G' = M \cap G'$. ■

In the case of almost-primitive groups we can even describe all the kernels of finitary linear images.

THEOREM 2.3. *The intransitive normal subgroup M of the almost-primitive finitary permutation group G is the kernel of a finitary linear representation of G if and only if the center of $G'/G' \cap M$ is elementary-abelian.*

Proof. Suppose first that there exists a K -vector space V such that $\overline{G} = G/M$ is a group of finitary transformations of V . From Schur's Lemma, the central section $G \cap \tilde{M}/M$ acts trivially on every infinite-dimensional \overline{G} -composition factor in V . Moreover, since infinite alternating groups contain finite p -groups of arbitrarily large solubility lengths, they cannot occur as sections of linear groups [16, 9.20 and 9.4], and thus G' is the smallest normal subgroup in G with a linear quotient. In particular, $\overline{G'}$ acts trivially on every finite-dimensional \overline{G} -composition factor in V . The intersection $\overline{N} = (\tilde{M} \cap G')M/M$ must then be contained in the unipotent radical of \overline{G} , whence \overline{N} is a central and unipotent subgroup of the finitary linear group $\overline{G'}$. Since \overline{N} corresponds to the center $\tilde{M} \cap G'/M \cap G'$ of $G'/M \cap G'$ under the canonical isomorphism $\overline{G'} \cong G'/M \cap G'$, it remains to show that \overline{N} is elementary-abelian.

Now $[V, \overline{h}]$ is a finite-dimensional \overline{G} -invariant subspace of V for every $h \in N$. Therefore $[V, \overline{N}]$ has an ascending \overline{G} -series with finite-dimensional factors. But the perfect almost-primitive group G' admits no proper linear quotient. Thus an easy induction along the ascending series ensures that $[V, \overline{N}, \overline{G'}] = 0$. In particular, \overline{N} stabilizes a series of length at most two in V . And so the periodic group \overline{N} must be elementary-abelian.

Conversely, if the finite central section $\overline{N} = (\tilde{M} \cap G')M/M$ of \overline{G} is elementary-abelian, then \overline{G} is a subdirect product of the quotients G/K , where K ranges over all subgroups of index p in $\tilde{M} \cap G'M$ which contain M . We may therefore assume without loss that \overline{N} has order p . Suppose for a moment that we can produce a finitary linear representation φ of G satisfying $N \cap \ker \varphi = M$. Then \overline{G} is a subdirect product of the finitary linear groups $G/G'M$, $G\tilde{M}/\tilde{M}$, and $G/\ker \varphi$. This shows that it suffices to find φ .

To this end, we recall the notation introduced at the beginning of this section and identify F with a fixed component in the base group $B = F^{(\Sigma)}$ of $W = F \operatorname{wr}_{\Sigma} \operatorname{FSym}(\Sigma)$. From replacing F by $F/(F \cap \tilde{M}' \cdot O^p(\tilde{M}))$ we may assume that \tilde{M} is an abelian p -group. There exists a chief factor X/Y in $F \cap \tilde{M}$ such that

$$N \cap MY^{(\Sigma)} = M < N = N \cap MX^{(\Sigma)}.$$

From replacing F by F/Y we may assume that Y is trivial. The Krasner–Kaloujnine embedding $F \rightarrow X \operatorname{wr}(F/X)$ induces a canonical embedding,

$$W \rightarrow X \operatorname{wr}(F/X) \operatorname{wr}_{\Sigma} \operatorname{FSym}(\Sigma) \leq X \operatorname{wr}_{\Delta} \operatorname{FSym}(\Delta),$$

where $\Delta = (F/X) \times \Sigma$. Note that X is cyclic of order p , so that $X^{(\Delta)}$ can be viewed as a vector space over the field $\operatorname{GF}(p)$. Since W acts transitively and hence almost-primitively on $X \times \Delta$, the normal subgroup M of W is even normal in $X \operatorname{wr}_{\Delta} \operatorname{FSym}(\Delta)$. We may thus consider the action of G by conjugation on $U = X^{(\Delta)}M/M$. Note that the subgroup N of $X^{(\Delta)}M$ acts trivially here. Because the wreath product is a finitary permutation group, G acts as a group of finitary transformations on U . Let $\pi: G \rightarrow \operatorname{FSym}(\Delta)$ denote the canonical projection of G into the top group of $X \operatorname{wr}_{\Delta} \operatorname{FSym}(\Delta)$, and define $f_g \in X^{(\Delta)}$ via $g = g\pi \cdot f_g$ for every $g \in G$. Let R (resp. S) be a left transversal of M in N (resp. of N in G), with $R \subseteq X^{(\Delta)}$. Put $T = S \cdot R$. We now extend the conjugation action of G on U to a finitary linear representation φ on the $\operatorname{GF}(p)$ -vector space $V = U \oplus \langle v \rangle$ via

$$[v, g\varphi] = f_t M \in U$$

whenever $g = tm$ with $t \in T$, $m \in M$. Clearly, $M \leq \ker \varphi$. Conversely, if $g = rm \in N \cap \ker \varphi$ with $r \in R$, $m \in M$, then $r \in X^{(\Delta)}$ gives $r\pi = 1$, while $g \in \ker \varphi$ implies $f_r \in M$. It follows that $N \cap \ker \varphi = M$, as desired. ■

Let us finally point out that, in contrast to the situation for totally imprimitive finitary permutation groups, almost-primitive finitary permutation groups admit only a few finitary permutation representations.

PROPOSITION 2.4. *Every infinite primitive finitary permutation group admits just one transitive finitary permutation representation on an infinite set, namely the natural one.*

Proof. Let $\operatorname{Alt}(\Sigma) \leq G \leq \operatorname{FSym}(\Sigma)$, and suppose that G acts as a transitive finitary permutation group on the infinite set Γ . Since G' is infinite and simple, we have $G' = \operatorname{Alt}(\Gamma)$ too. From [1, Corollary 3.5], the complex vector space $V = [\mathbb{C}\Gamma, G]$ is an irreducible $\mathbb{C}G$ -module. Hence [1, Theorems A and B] yield $V \cong_{\mathbb{C}G} [\mathbb{C}\Sigma, G]$. (Note that for alternating groups the results from [1] used above can also be found in [4].)

For every $g \in G$ there is some $\sigma_g \in \Sigma \setminus \text{supp}_\Sigma g$. The action of g on the basis $\{\sigma - \sigma_g \mid \sigma \in \Sigma - \{\sigma_g\}\}$ of V is similar to the action of g on $\Sigma - \{\sigma_g\}$. Consider the decomposition of g into a product of disjoint cycles in $\text{FSym}(\Sigma)$. Each n -cycle gives rise to an occurrence with multiplicity one of every n th root $\neq 1$ of unity in the Jordan normal form of the transformation g of V . The above consideration holds likewise with Γ in place of Σ . This shows that the cycle structure of g is independent of whether we consider g as an element in $\text{FSym}(\Sigma)$ or in $\text{FSym}(\Gamma)$.

We aim to show next that, for every finite subset Σ_0 of Σ , the subgroup $\text{Alt}(\Sigma_0)$ of G acts naturally on Γ . Assume that this fails for a certain $\Sigma_0 = \{\sigma_1, \dots, \sigma_{n+1}\}$ of smallest possible size. Clearly $n \geq 3$. Now $X = \text{Alt}(\{\sigma_1, \dots, \sigma_n\})$ acts naturally on its support $\{\gamma_1, \dots, \gamma_n\}$ in Γ . And the 3-cycle $y = (n-1 \ n \ n+1)$ in $Y = \text{Alt}(\Sigma_0)$ must move an additional symbol $\gamma_{n+1} \in \Gamma$. If $y = (\gamma_i \ \gamma_{n+1} \ \gamma_{n+2})$ for some $i \leq n$ and $\gamma_{n+2} \notin \Gamma_0 = \{\gamma_1, \dots, \gamma_{n+1}\}$, then y would commute with the stabilizer of the point γ_i in X . This could only happen when $n = 3$. But then $(\gamma_1 \ \gamma_2 \ \gamma_3) \cdot y$ would be an element of order 5 in the alternating group Y of degree 4. This shows that Γ_0 is the support of $Y = \langle X, y \rangle$ in Γ . But now $Y = \text{Alt}(\Gamma_0)$ acts on Γ_0 with point stabilizer $X = \text{Alt}(\Gamma_0 \setminus \{\gamma_{n+1}\})$. This action is similar to the action of Y by right multiplication on the set $[X \setminus Y]$ of right cosets of X in Y , and hence is similar to the natural action of Y . This contradicts the initial assumption on Σ_0 .

Consider three 3-cycles $(\sigma_1 \ \sigma_2 \ \sigma_3)$, $(\sigma_1 \ \sigma_4 \ \sigma_5)$, $(\sigma_1 \ \sigma_6 \ \sigma_7)$ in $\text{Alt}(\Sigma)$, where $\sigma_1, \dots, \sigma_7$ are supposed to be distinct. These 3-cycles are contained in a finite alternating subgroup of G , which is represented naturally on its support in Γ . The first two 3-cycles have a unique common $\gamma_1 \in \Gamma$ in their support, which must also show up in the support of the third 3-cycle, and hence of any 3-cycle moving σ_1 . For any 3-cycle g in G , the centralizer $C_G(g)$ is the direct product of $\langle g \rangle$ with the pointwise stabilizer in G of the support of g in Σ , or likewise in Γ . In particular, the stabilizer H in G of σ_1 can be written as the union over all $O^3(C_G(g))$, where g ranges over all 3-cycles moving σ_1 . And this union is contained in the stabilizer of γ_1 . Both stabilizers are maximal subgroups in G , so that they are equal. Now the actions of G on Σ and on Γ are both natural, because they are both similar to the action of G on $[H \setminus G]$. ■

THEOREM 2.5. *Every almost-primitive finitary permutation group has just finitely many inequivalent transitive finitary permutation representations.*

Proof. We shall use the notation introduced at the beginning of this section. Let K be the point stabilizer of a transitive finitary permutation representation of G . We may assume that this is the action of G by right multiplication on the set $[K \setminus G]$ of right cosets of K in G . Since G has just finitely many finite quotients, we only need to consider the case where

$[K \backslash G]$ is infinite. Then the kernel of the action of G on $[K \backslash G]$ is intransitive and is contained in B . Now $G' = W' = D \rtimes \text{Alt}(\Sigma)$, where D denotes the set of all base group functions $f: \Sigma \rightarrow F$ in B , for which $\prod_{\sigma \in \Sigma} (\sigma)f \in F'$ (see [8, Corollary 1.4.9]). In particular, $\text{Alt}(\Sigma) \leq G$, and the $(B \cap G)$ -orbits in $[K \backslash G]$ are maximal G -blocks. From Proposition 2.4, the group $G/B \cap G$ acts naturally on the system of imprimitivity formed by these blocks. Therefore $K \leq G_{\{\Delta\}}$ for some $(B \cap G)$ -orbit $\Delta = F \times \{\sigma\}$ in $\Omega = F \times \Sigma$.

We aim to show next that $[G_{\Delta}, G_{\Delta}] \leq K$. To this end, let N denote the core of K in $G_{\{\Delta\}}$, and consider elements $a, b \in G_{\Delta}$. Choose a transversal T of N in $G_{\{\Delta\}}$. Since $\text{Alt}(\Sigma)$ acts highly transitively on Σ , there exists $g \in \text{Alt}(\Sigma \setminus \{\sigma\})$ such that $\text{supp}_{\Omega} T^g \cap \text{supp}_{\Omega} b = \emptyset$. Now T^g is a transversal of N in $G_{\{\Delta\}}$ too, and so $a = t^g h$ for certain $h \in N$, $t \in T$. It follows that

$$[a, b] = [t^g h, b] = [t^g, b]^h [h, b] = [h, b] \in N \leq K.$$

It remains to show that $[G_{\Delta}, G_{\Delta}]$ has finite index in $G_{\{\Delta\}}$; up to conjugacy there are just finitely many choices for K then. However, because $\text{Alt}(\Sigma)$ acts highly transitively on Σ , it is straightforward that $[G_{\Delta}, G_{\Delta}] = D_{\Delta} \rtimes \text{Alt}(\Sigma \setminus \{\sigma\})$, whence

$$\begin{aligned} |G_{\{\Delta\}} : [G_{\Delta}, G_{\Delta}]| &\leq |(B \rtimes \text{FSym}(\Sigma \setminus \{\sigma\})) : (D_{\Delta} \rtimes \text{Alt}(\Sigma \setminus \{\sigma\}))| \\ &= 2 \cdot |B : D_{\Delta}| = 2 \cdot |F| \cdot |B_{\Delta} : D_{\Delta}| \leq 2 \cdot |F|^2 < \infty. \end{aligned}$$

■

3. POINT STABILIZERS IN TOTALLY IMPRIMITIVE GROUPS

In the remaining parts of this paper, G will always denote a totally imprimitive subgroup of $\text{FSym}(\Omega)$, that is, a transitive subgroup of $\text{FSym}(\Omega)$ such that every finite subset of the infinite set Ω is contained in a finite G -block. The proof of Theorem A will require the following two technical lemmata.

LEMMA 3.1. *Let G be a totally imprimitive subgroup of $\text{FSym}(\Omega)$, and let Δ be a finite subset of Ω . Then G_{Δ} is the point stabilizer of a transitive finitary permutation representation of G .*

Proof. Let $\Delta = \{\delta_1, \dots, \delta_n\}$, and consider the diagonal action of G on the cartesian power Ω^n . Let Γ be the G -orbit of the tuple $(\delta_1, \dots, \delta_n)$ in Ω^n . Clearly G_{Δ} is the point stabilizer in G with respect to the action on Γ . It therefore suffices to show that G acts finitarily on Γ . Since G is a totally imprimitive subgroup of $\text{FSym}(\Omega)$, the finite set Δ is contained in a finite G -block $\Lambda \subset \Omega$. It follows that Γ is contained in $\Sigma = \bigcup_{t \in G} (\Lambda^t)^n$. But every $g \in G$ fixes all but finitely many G -translates of Λ pointwise, so that g acts finitarily on Σ too. ■

LEMMA 3.2. *Let G be a transitive subgroup of $\text{Sym}(\Omega)$ with point stabilizer H , and let T be a right transversal of H in G . Then $g \in G$ acts finitarily on Ω if and only if $g^{t^{-1}} \in H$ for all but finitely many $t \in T$.*

Proof. The action of G on Ω is equivalent to the action of G on the set $[H \backslash G]$ of right cosets of H in G via right multiplication. And so the element $g \in G$ acts finitarily on Ω if and only if $Htg = Ht$ for all but finitely many $t \in T$. But this condition is equivalent to $g^{t^{-1}} \in H$ for all but finitely many $t \in T$. ■

Remark 3.3. Condition (A.2) in Theorem A is equivalent to

(A.2') For every finite subgroup F of G there exists a finite G -block Δ_F in Ω such that $\langle F^G \rangle \cap G_{\Delta_F} \leq K$.

Here Δ_F can always be chosen such that $\Delta \subseteq \Delta_F$; then K has a finite index in $G_{\{\Delta_F\}}$ too.

Proof. Suppose first that K contains all but finitely many conjugates of the finite subgroup F of G . Since the intransitive normal subgroup $\langle F^G \rangle$ of G has finite orbits in Ω , there exists a finite G -block $\Delta_F \supseteq \Delta$ in Ω which is the union of $\langle F^G \rangle$ -orbits and contains the supports of all those finitely many G -conjugates of F , which lie outside K . Now $\langle F^G \rangle \cap G_{\Delta_F} \leq K$.

Suppose next that there exists a finite G -block Δ_F in Ω such that $\langle F^G \rangle \cap G_{\Delta_F} \leq K$. We may assume that Δ_F contains the support of F , and that Δ_F is a union of $\langle F^G \rangle$ -orbits. Then the supports of all but finitely many G -conjugates of F are disjoint from Δ_F , whence these conjugates are contained in $\langle F^G \rangle \cap G_{\Delta_F}$. ■

Proof of Theorem A. Suppose first that conditions (A.1) and (A.2) are satisfied. We aim to show that G acts finitarily on $[K \backslash G]$ via right multiplication. Consider a fixed element g in G . From (A.2') there exists a finite G -block $\Delta^* \supseteq \Delta$ in Ω such that $\langle g^G \rangle \cap G_{\Delta^*} \leq K$. Because $G_{\{\Delta\}}$ has a finite index in $G_{\{\Delta^*\}}$, we may replace Δ by Δ^* without losing condition (A.1).

Let S (resp. T) denote a right transversal of K in $G_{\{\Delta\}}$ (resp. of $G_{\{\Delta\}}$ in G) with $1 \in T$. Note that $S \cdot T$ is a right transversal of K in G . Since T is also part of a right transversal of G_Δ in G , Lemmata 3.1 and 3.2 ensure that $g^{t^{-1}} \in G_\Delta$ for all but finitely many $t \in T$. But $G_{\{\Delta\}}$ normalizes G_Δ , and so

$$g^{(st)^{-1}} = g^{t^{-1}s^{-1}} \in (G_\Delta)^{s^{-1}} \cap \langle g^G \rangle = G_\Delta \cap \langle g^G \rangle \leq K$$

for all $s \in S$ and all but finitely many $t \in T$. However, S is finite, whence $g^{(st)^{-1}} \notin K$ for at most those finitely many $s \cdot t$, for which $g^{t^{-1}} \notin G_\Delta$. Now Lemma 3.2 yields that g acts finitarily on $[K \backslash G]$.

Conversely, let K be the point stabilizer and M be the kernel of a totally imprimitive finitary permutation representation of G on the infinite set Σ

$= [K \setminus G]$. Then K/M is a subdirect product of finite groups K_j ($j \in \omega$). Let $\{L_i\}_{i \in \omega}$ be a tower of finite subgroups of K with union K . Choose another such tower $\{F_j\}_{j \in \omega}$, such that the image of the projection of K/M onto $K_0 \times \cdots \times K_j$ equals the image of the projection of $F_j M/M$ onto $K_0 \times \cdots \times K_j$. For each i there exists j_i such that $L_i M/M$ projects trivially onto $K_{j_i+1} \times K_{j_i+2} \times \cdots$. Therefore $[L_i, L_i]M/M$ is contained in $[L_i, F_{j_i}^g]M/M$, no matter how we choose $g \in K$.

Next, let Γ be a K -orbit in Ω . Assume that Γ is infinite. Let N denote the kernel of the action of K on Γ . Since K acts finitarily, there is $g \in K$ such that $[L_i, F_{j_i}^g] \leq N$ (see [9, Lemma 3.2]). It follows that $[K, K] = \bigcup_{i \in \omega} [L_i, L_i] \leq N \cdot M$ and $(K/N)' \leq MN/N$. But infinity of Σ implies that G' is not contained in M . Thus M acts intransitively on Ω . Hence $\exp M < \infty$ and $\exp(K/N)' < \infty$, in contradiction to the infinity of Γ (see [9, Theorem 1]). This contradiction shows that Γ is finite. In particular, Γ is contained in a finite G -block $\Delta \subset \Omega$, whence $K \leq G_{\{\Delta\}}$. The right cosets of $G_{\{\Delta\}}$ form a system of imprimitivity with respect to the action of G on $[K \setminus G]$. Hence K must have a finite index in $G_{\{\Delta\}}$. It remains to verify condition (A.2').

To this end, let F be a finite subgroup of G , and let $J = \langle F^G \rangle$. Choose a finite G -block $\Delta_1 \subseteq \Omega$ containing $\Delta \cup \text{supp}_\Omega F$. Then $J \leq G_{\{\Delta_1\}}$ and $|J : J \cap K| \leq |G_{\{\Delta_1\}} : K| < \infty$. From enlarging F slightly we may assume that $J = F(J \cap K)$. There also exists a finite G -block Λ_1 in Σ containing $\text{supp}_\Sigma F$. Now $G_{\{\Lambda_1\}}$ is the point stabilizer of the totally imprimitive action of G on the set of all G -translates of Λ_1 . It therefore follows as above that $G_{\{\Lambda_1\}}$ satisfies (A.1) in place of K , that is, $G_{\{\Lambda_1\}}$ is a subgroup of finite index in $G_{\{\Delta_2\}}$ for some finite G -block $\Delta_2 \supseteq \Delta_1$ in Ω .

Let S be a right transversal of $G_{\{\Delta_2\}}$ in G with $1 \in S$, and consider $J_0 = \langle F^g \mid g \in G_{\{\Delta_2\}} \rangle$. Then $J = \text{Dr}_{s \in S} J_0^s$, where $\text{supp}_\Omega J_0^s \subseteq \Delta_2 s$ for all $s \in S$. Since $G_{\{\Lambda_1\}}$ is contained in $G_{\{\Delta_2\}}$, the set of all cosets of K in G , which are contained in $G_{\{\Delta_2\}}$, form a finite G -block $\Lambda_2 \supseteq \Lambda_1$ in Σ , and $G_{\{\Lambda_2\}} = G_{\{\Delta_2\}}$. Moreover, $\text{supp}_\Sigma J_0^s \subseteq \Lambda_2 s$ for all $s \in S$. Therefore

$$J \cap G_{\Delta_2} = \text{Dr}_{1 \neq s \in S} J_0^s = J \cap G_{\Lambda_2} \leq K.$$

■

It is evident from the conditions in Theorem A that every intersection of finitely many point stabilizers of various totally imprimitive finitary permutation representations of G is again the point stabilizer of a totally imprimitive finitary permutation representation of G . The following observation will be useful in the proof of Theorem B.

PROPOSITION 3.4. *Let G be a totally imprimitive subgroup of $\text{FSym}(\Omega)$. If K is the point stabilizer of some totally imprimitive finitary permutation*

representation of G , then the finite G -block Δ in Theorem A may be so chosen that $[G_\Delta, G_\Delta] \leq K$.

Proof. Choose Δ as in Theorem A, and let N denote the core of K in $G_{\{\Delta\}}$. Consider a transversal T of N in $G_{\{\Delta\}}$, and choose a finite G -block Δ^* in Ω which contains Δ and $\text{supp}_\Omega T$. Since $G_{\{\Delta\}}$ is a subgroup of finite index in $G_{\{\Delta^*\}}$, condition (A.1) holds with Δ^* in place of Δ . Moreover, given any two elements $a, b \in G_{\Delta^*}$, there exist $t \in T$ and $h \in N$ such that $a = th$, whence

$$[a, b] = [th, b] = [t, b]^h [h, b] = [h, b] \in N.$$

This shows that $[G_{\Delta^*}, G_{\Delta^*}] \leq N \leq K$, and so we may replace Δ by Δ^* . ■

Proof of Theorem B. If conditions (B.1) and (B.2) are satisfied, then H has 2^{\aleph_0} subgroups of finite index which contain N . From Theorem A, all these subgroups are point stabilizers of faithful totally imprimitive finitary permutation representations of G . Since G is countable, we have thus found 2^{\aleph_0} such inequivalent representations.

Conversely, suppose that G has uncountably many inequivalent totally imprimitive finitary permutation representations. Then there exist uncountably many subgroups K in G which satisfy the conditions in Theorem A and in Proposition 3.4. Since Ω is countable, there are just countably many finite G -blocks in Ω . Hence there is a finite G -block Δ such that $G_{\{\Delta\}}$ contains uncountably many subgroups K of finite index, which satisfy condition (A.2) and which contain $[G_\Delta, G_\Delta]$. Since G_Δ has a finite index in $G_{\{\Delta\}}$, every subgroup of finite index in G_Δ is contained in just finitely many different subgroups of $G_{\{\Delta\}}$. From Lemma 3.2 it now follows that the intersections $K \cap G_\Delta$ form an uncountable family \mathcal{L}_0 of subgroups of finite index G_Δ , which satisfy condition (A.2) and which contain $[G_\Delta, G_\Delta]$. Let N_0 denote the intersection of all groups in \mathcal{L}_0 . Note that G_Δ/N_0 is infinite and abelian. In fact, we may also assume that all groups from \mathcal{L}_0 have the same index in G_Δ . Then G_Δ/N_0 has a finite exponent. The groups L/N_0 ($L \in \mathcal{L}_0$) have uncountably many different p -parts for some fixed prime p . From replacing each L by its product with the preimage of the p' -part of G_Δ/N_0 we may assume that G_Δ/N_0 is an infinite abelian p -group of finite exponent.

Next, let $\{F_i\}_{i \in \omega}$ be an ascending chain of finite subgroups with union G . Without loss, $F_0 \cap G_\Delta \not\leq N_0$. However, some intersection L_0 of finitely many groups in \mathcal{L}_0 satisfies $L_0 \cap F_0 \leq N_0$. As above we may replace every group in \mathcal{L}_0 by its intersection with L_0 . Since there are just countably many finite G -blocks in Ω , there are a finite G -block $\Delta_0 \supseteq \Delta$ and an uncountable subfamily \mathcal{L}_1 of \mathcal{L}_0 such that $\langle F_0^G \rangle \cap G_{\Delta_0}$ is contained in each group from \mathcal{L}_1 . Let N_1 denote the intersection of all groups in \mathcal{L}_1 . Again G_Δ/N_1 is infinite.

Iteration of the above argument produces a tower $\{\Delta_i\}_{i \in \omega}$ of finite G -blocks and an ascending chain of subgroups $\{N_i\}_{i \in \omega}$ in G_Δ such that

- (1) N_{i+1} contains $\langle F_i^G \rangle \cap G_{\Delta_i}$ for all i ,
- (2) $F_i \cap G_\Delta \not\leq N_i(F_{i-1} \cap G_\Delta)$ for all i , and
- (3) $N_{i+1} \cap F_i \leq N_i$ for all i .

Let $N = \bigcup_{i \in \omega} N_i$. As in Remark 3.3, property (1) implies that N satisfies (B.2). Property (3) yields $F_i \cap N_j = F_i \cap N_i$ for all $j \geq i$, and thus $F_i \cap N = F_i \cap N_i$ for all i . Assume that $(F_i \cap G_\Delta)N = (F_{i-1} \cap G_\Delta)N$ for some i . Then

$$\begin{aligned} F_i \cap G_\Delta &\leq F_i \cap N(F_{i-1} \cap G_\Delta) \\ &= (F_i \cap N)(F_{i-1} \cap G_\Delta) \leq N_i(F_{i-1} \cap G_\Delta). \end{aligned}$$

This contradiction to property (2) shows that $(F_{i-1} \cap G_\Delta)N/N$ is a proper subgroup of $(F_i \cap G_\Delta)N/N$ for all i . In particular, G_Δ/N is an infinite abelian p -group of finite exponent.

Let G_Δ/X denote the largest elementary-abelian p -quotient in G_Δ . Then G_Δ is a subgroup of finite index in H , and X is normal in H . Choose a finite subgroup E in H such that $H = EG_\Delta$. Since $\text{supp}_\Omega E$ is finite, $\langle E^H \rangle$ is finite too. Hence H/X is the extension of the finite normal subgroup $Y/X = \langle E^H \rangle X/X$ by an infinite elementary-abelian p -group. It follows that the quotient $H/NY = G_\Delta Y/NY$ satisfies (B.1). And clearly NY satisfies (B.2). ■

4. TOTALLY IMPRIMITIVE REPRESENTATIONS OF ITERATED WREATH PRODUCTS

The content of Theorem B will now be illustrated with an application to infinite iterated wreath products $W = \text{wr}_{n \in \omega} F_n = F_0 \text{ wr}_{\Lambda_1} F_1 \text{ wr}_{\Lambda_2} F_2 \text{ wr} \cdots$ of finite transitive permutation groups $F_n \leq \text{Sym}(\Lambda_n)$. Such a wreath product W is the direct limit of the iterated wreath products $W_0 = F_0$ and

$$W_n = W_{n-1} \text{ wr}_{\Lambda_n} F_n = F_0 \text{ wr}_{\Lambda_1} F_1 \text{ wr} \cdots \text{ wr}_{\Lambda_n} F_n \quad (n \geq 1),$$

with respect to the canonical embedding of W_{n-1} onto a fixed component of the base group of W_n (corresponding to $\nu_n \in \Lambda_n$). As in [6], we shall consider W as a totally imprimitive finitary permutation group on the set

$$\Omega = \{(\lambda_n)_{n \in \omega} \in \prod_{n \in \omega} \Lambda_n \mid \lambda_n = \nu_n \text{ for all but finitely many } n \in \omega\}$$

via $(\lambda_n)x = (\lambda'_n)$ for $x \in F_i$ where

$$\lambda'_n = \begin{cases} \lambda_n x & \text{if } n = i \text{ and } \lambda_m = \nu_m \text{ for all } m > n, \\ \lambda_n & \text{else.} \end{cases}$$

For each n , the set $\Delta_n = \text{supp}_\Omega W_n = \{(\lambda_m) \in \Omega \mid \lambda_m = \nu_m \text{ for all } m > n\}$ is a finite W -block in Ω . To achieve a uniform notation, let $\Delta_{-1} = \{(\nu_n)_{n \in \omega}\}$ and $W_{-1} = 1$.

Consider the point stabilizer $H = W_{\Delta_{-1}}$ in W . Clearly $H = \text{Dr}_{n \in \omega} H_n$ where H_n denotes the pointwise stabilizer of Δ_{n-1} in W_n . Note that H_n is isomorphic to $W_{n-1} \text{ wr}_{\Lambda_n \setminus \{\nu_n\}} E_n$, where

E_n denotes the stabilizer of ν_n in F_n .

Therefore H_n/H'_n is a direct product of a positive number of copies of F_i/F'_i ($0 \leq i \leq n-1$) and a copy of E_n/E'_n (see [8, Theorem 1.4.8]). Here the copies of F_i/F'_i are induced from the i th layer of the wreath product W_n .

THEOREM 4.1. *The infinite iterated wreath product $W = F_0 \text{ wr}_{\Lambda_1} F_1 \text{ wr}_{\Lambda_2} F_2 \text{ wr} \cdots$ of finite transitive permutation groups $F_n \leq \text{Sym}(\Lambda_n)$ has uncountably many inequivalent totally imprimitive finitary permutation representations if and only if there exists a prime which divides $|F_n/F'_n| \cdot |E_n/E'_n|$ for infinitely many n .*

Proof. We use the notation introduced above. Suppose first that there exists a prime p dividing $|F_n/F'_n| \cdot |E_n/E'_n|$ for infinitely many n . Let $N = H'_0 \times \text{Dr}_{n \geq 1} N_n$, where $N_n = H'_n(H_n \cap \langle W_{n-2}^W \rangle)$. For every n , the subgroup N of H contains every W -conjugate of W_n whose support is contained in $\Omega \setminus \Delta_{n+1}$. Moreover, H/N contains a copy of $\text{Dr}_{n \geq 1} (F_{n-1}/F'_{n-1} \times E_n/E'_n)$. In particular, H/N has an infinite elementary-abelian p -quotient. Now Theorem B yields 2^{\aleph_0} inequivalent faithful totally imprimitive finitary permutation representations for W .

Conversely, suppose that W satisfies (B.1) and (B.2). Condition (B.2) ensures that there exists an increasing and unbounded sequence $\{i_n\}_{n \in \omega}$ of natural numbers $i_n \leq n-1$ such that N contains $H_n \cap \langle W_{i_n}^W \rangle$ for all n . Therefore the elementary-abelian p -section W_Δ/N can only be infinite when p divides $|F_n/F'_n|$ or $|E_n/E'_n|$ for infinitely many n . ■

Theorem 4.1 implies, for example, that the iterated wreath product of infinitely many finite alternating groups of degrees ≥ 6 in their natural action has just countably many totally imprimitive finitary permutation representations, while an iterated wreath product of infinitely many copies of the alternating groups of degree 5 in its natural action admits uncountably many totally imprimitive finitary permutation representations.

Note also that there is no relation at all between the number of totally imprimitive finitary permutation representations of a given totally imprimitive finitary permutation group and those of its transitive subgroups. To see this, consider first the iterated wreath product $\text{wr}_{n \in \omega} \text{Sym}(d_n)$ of infinitely many finite symmetric groups of degrees $d_n \geq 6$. This group has uncountably many totally imprimitive finitary permutation representations, but it

contains the transitive subgroup $\text{wr}_{n \in \omega} \text{Alt}(d_n)$, which admits just countably many totally imprimitive finitary permutation representations. Conversely, the affine special linear group $\text{ASL}(m, p)$ (p prime) in its natural action is a semidirect product of a regular elementary-abelian subgroup of order p^m by the special linear group $\text{SL}(m, p)$, which constitutes the point stabilizer. Here $\text{SL}(m, p)$ and $\text{ASL}(m, p)$ are perfect provided $m > 2$, or $m = 2$ and $p > 3$. Hence, for such values of m_n and p , the group $\text{wr}_{n \in \omega} \text{ASL}(m_n, p)$ has just countably many totally imprimitive finitary permutation representations, but it contains the wreath product of elementary-abelian groups of orders p^{m_n} as transitive subgroup, which of course admits uncountably many totally imprimitive finitary permutation representations.

However, there are situations where a transitive subgroup G of an iterated wreath product W is related more closely to W than in the above examples. Namely, we say that $W = \text{wr}_{n \in \omega} F_n$ is an *enveloping wreath product* for G if G is a transitive subgroup of W such that $W_{\{\Delta_n\}} = G_{\{\Delta_n\}} W_{\Delta_n} \langle W_{n-1}^W \rangle$ for all n . Note that every totally imprimitive finitary permutation group G is contained in an enveloping wreath product W with primitive factors F_n . If F_n is nonabelian, then maximality of the point stabilizer E_n implies that F_n/F'_n occurs as a quotient of E_n/E'_n . Therefore we only consider the abelianization of E_n in the following handy criterion.

THEOREM 4.2. *Let $W = \text{wr}_{n \in \omega} F_n$ be an enveloping wreath product for the totally imprimitive finitary permutation group G with primitive factors $F_n \leq \text{Sym}(\Lambda_n)$. Suppose that there exists a prime p which divides $|E_n/E'_n|$ for infinitely many n . Then G has uncountably many inequivalent totally imprimitive finitary permutation representations.*

Proof. We stick to the notation introduced at the beginning of this section, and let $B_n = \langle W_n^W \rangle$ for all n . The normal subgroup $L = \prod_{n \in \omega} (G \cap B_n \cap W_{\Delta_n})$ of the point stabilizer $K = H \cap G$ contains almost all G -conjugates of every finite subgroup of G . Assume that the largest elementary-abelian p -quotient K/N of K/L is finite. Then there exists a finite subgroup Y in G such that $K = YN$. Choose n such that $Y \leq B_{n-1}$, and such that p divides $|E_n/E'_n|$. Since W is an enveloping wreath product for G , we have $E_n W_{\Delta_n} B_{n-1} = H B_{n-1} = K W_{\Delta_n} B_{n-1}$, and there exists a finite subgroup X in K such that $X W_{\Delta_n} B_{n-1}$ is a normal subgroup of index p in $H B_{n-1}$. But now $K/K \cap X W_{\Delta_n} B_{n-1} \cong H B_{n-1}/X W_{\Delta_n} B_{n-1}$, so that $K = YN \leq X W_{\Delta_n} B_{n-1}$. This contradiction shows that K/N is infinite, whence G has uncountably many inequivalent totally imprimitive finitary permutation representations by Theorem B. ■

In Theorem 4.2, we cannot remove the assumption that E_n is nontrivial for infinitely many n . This is shown by the following example.

EXAMPLE 4.3. The infinite wreath product $W = \text{wr}_{n \in \omega} F_n$ of cyclic groups $F_{2i} \cong C_p$ of fixed prime order p and of alternating groups $F_{2i+1} \cong \text{Alt}(d_i)$ of degrees $d_i \geq 6$ in their natural action is an enveloping wreath product for a transitive subgroup G which admits just countably many totally imprimitive permutation representations.

Proof. Again we shall use the notation introduced at the beginning of this section. Let $G = \bigcup_{n \in \omega} G_n$, where $G_0 = 1$, and where G_{n+1} is the derived subgroup of $G_n \text{ wr } C_p \text{ wr } \text{Alt}(d_n) \leq W_{2n+1}$.

Suppose by induction that G_n is perfect. Then G_{n+1} is the subgroup of $(G_n \text{ wr } C_p) \text{ wr } \text{Alt}(d_n)$ generated by G_n , by $\text{Alt}(d_n)$, and by all base group functions $f: \{1, \dots, d_n\} \rightarrow C_p$ for which $\prod_i (i)f = 1$ see [8, Corollary 1.4.9]). Since the alternating group is 2-transitive, it follows that G_{n+1} is perfect. We have thus shown that G_n is perfect for all n . From the above structure of G_n it is obvious that W is an enveloping wreath product for G .

We shall conclude by showing that each $H \cap G_n$ is perfect. Then the point stabilizer in G is perfect, whence G has just countably many totally imprimitive finitary representations by Theorem B. However, $H \cap G_{n+1}$ is generated by $\text{Alt}(\{2, \dots, d_n\})$, by those base group functions as above, for which in addition $(1)f = 1$, and by $H \cap \langle G_n^{G_{n+1}} \rangle$. It follows that $H \cap G_{n+1}$ is isomorphic to the direct product of $H \cap G_n$ with the derived subgroup of $G_n \text{ wr } C_p \text{ wr } \text{Alt}(d_n - 1)$. And as above, this latter group is perfect. Therefore induction ensures that $H \cap G_n$ is perfect too for each n . ■

In the above example we could as well replace the cyclic factors by wreath products of finitely many cyclic groups, since each such wreath product contains a regular abelian subgroup. At present it is not clear, however, whether a locally solvable totally imprimitive finitary permutation group, for which there exists a prime p such that the orders of the p -elements are unbounded, always has uncountably many totally imprimitive finitary permutation representations. Evidently, this holds for locally solvable π -groups, where π is a finite set of primes (Theorem B), and for locally solvable iterated wreath products (Theorem 4.1).

The relationship between a totally imprimitive permutation group G and its enveloping wreath product W becomes slightly more satisfactory when G is dense in W with respect to the topology of pointwise convergence, that is, when $W_{\{\Delta_n\}} = G_{\{\Delta_n\}} W_{\Delta_n}$ for all n (see Proposition 5.3). But even in this case the enveloping wreath product W of a totally imprimitive permutation group G with “many” totally imprimitive permutation representations can have “few” totally imprimitive permutation representations.

EXAMPLE 4.4. The derived subgroup of the infinite wreath product $W = \text{wr}_{n \in \omega} F_n$ of a cyclic group F_0 of prime order and of alternating groups

$F_n \cong \text{Alt}(d_n)$ of degrees $d_n \geq 6$ in their natural action has uncountably many totally imprimitive permutation representations.

Proof. Once again we stick to the notation introduced at the beginning of this section. For each n , let $X_n = W' \cap B \cap W_n$, where $B = \langle F_0^W \rangle$. Moreover, let N_n be the join of all W -conjugates of X_n whose support is disjoint from Δ_n . Finally, N shall denote the semidirect product $\langle N_n \mid n \in \omega \rangle \rtimes (H \cap Q)$, where Q is the derived subgroup of $\text{wr}_{n \geq 1} F_n$.

Since $H \cap W' = (H \cap W' \cap B) \rtimes (H \cap Q)$, the quotient $(H \cap W')/N$ is isomorphic to the elementary-abelian section $(H \cap W' \cap B)/(N \cap B)$ of B . Consider $Y_n = \{x \in X_n \mid \text{supp}_\Omega x \subseteq \Delta_n \setminus \Delta_{n-1}\}$ and $Z_n = Y_n \cap \langle X_{n-1}^W \rangle$. For the elements $a, b \in F_n \cong \text{Alt}(d_n)$ corresponding to (12)(45) resp. (23)(45), and for $w \in (B \cap W_{n-1}) \setminus X_{n-1}$, we have $[w^a, b] \in Y_n \setminus Z_n$. Moreover, $Y = \text{Dr}_{n \in \omega} Y_n \leq H \cap W' \cap B$ and $N \cap Y = \text{Dr}_{n \geq 1} Z_n$. Therefore $YN/N \cong Y_0 \times \text{Dr}_{n \geq 1} Y_n/Z_n$ is an infinite section in $(H \cap W')/N$. In view of Theorem B it remains to show that N contains almost all W -conjugates of every finite subgroup F of W' . But every such F is contained in $X_n Q$ for some n , and so every W -conjugate of F , whose support is disjoint from Δ_n , is indeed contained in $N_n(H \cap Q) \leq N$. ■

5. TOPOLOGICAL CONSIDERATIONS

Every intersection of finitely many point stabilizers of various totally imprimitive finitary permutation representations of the totally imprimitive group G is again the point stabilizer of a totally imprimitive finitary permutation representation of G . Therefore G becomes a topological group with respect to the topology in which the point stabilizers of all totally imprimitive finitary permutation representations form a system of open neighborhoods of the identity. This topology will be called the *representation topology* on G .

PROPOSITION 5.1. *Let G be a totally imprimitive finitary permutation group with the representation topology.*

(a) *Every proper open subgroup of G is the point stabilizer of a totally imprimitive finitary permutation representation of G .*

(b) *A proper normal subgroup M of G is closed if and only if G/M has a faithful permutation representation such that G/M acts as a totally imprimitive finitary permutation group on each orbit.*

Proof. (a) Every open subgroup U of G contains a point stabilizer K of some totally imprimitive finitary permutation representation of G on the set $[K \backslash G]$ via right multiplication. The cosets of U in G form a system of

imprimitivity, and U is the point stabilizer of the totally imprimitive action of G on this system of imprimitivity.

(b) If M is closed, then it is the intersection of all subgroups of the form MK where K is open in G . From (a), every such group MK is a point stabilizer of some totally imprimitive finitary permutation representation of G . ■

It follows from Proposition 5.2 and Example 6.4 that not every closed normal subgroup of G is the kernel of a finitary permutation representation of G .

QUESTION. *Under which circumstances are closed normal subgroups kernels of finitary permutation representations?*

There exists a natural topology on G which is weaker than the representation topology, and which is related to the original permutation representation of G on Ω . Namely, this is the topology in which the pointwise stabilizers of finite subsets of Ω form a system of open neighborhoods of the identity (see [3]). We shall refer to this topology as the *topology of pointwise convergence*.

In general, the topology of pointwise convergence is much weaker than the representation topology. This can be seen as follows. The topology of pointwise convergence has a countable basis of open neighborhoods of the identity, and every open subgroup U contains one member H of this basis. Since G acts finitarily on $[H \setminus G]$, the index of H in U is finite. It follows that there are just countably many open subgroups in G with respect to the topology of pointwise convergence. However, Theorem B provides examples of totally imprimitive finitary permutation groups with uncountably many point stabilizers.

Quite amazingly, the following proposition shows that closedness of normal subgroups is independent of which of the two topologies one considers.

PROPOSITION 5.2. *Let M be a normal subgroup of the totally imprimitive finitary permutation group G . The following are equivalent.*

- (1) G/M has a trivial center.
- (2) M is closed in G with respect to the topology of pointwise convergence.
- (3) M is closed in G with respect to the representation topology.

Proof. Suppose first that G/M is a proper quotient with a trivial center. Then M is intransitive. Let \bar{M} denote the closure of M in G with respect to the topology of pointwise convergence. Clearly $\bar{M} = \bigcap_{i \in \omega} MG_{\Delta_i}$ for a tower $\{\Delta_i\}_{i \in \omega}$ of finite M -invariant G -blocks with union Ω . Consider $h \in \bar{M}$ and $g \in G$. Choose j such that $\text{supp}_{\Omega} g \subseteq \Delta_j$. Since each element in

MG_{Δ_j} acts on Δ_j like an element from M , there exists $m \in M$ satisfying $[h, g] = [m, g] \in M$. This shows that the section \overline{M}/M is central in G . And so (2) follows from (1).

The topology of pointwise convergence is weaker than the representation topology.

Suppose finally that (3) holds, and assume that $[z, G] \leq M$ for some $z \in G \setminus M$. By Theorem 5.1(b) there exists a kernel $N \geq M$ of some totally imprimitive finitary permutation representation of G such that $z \in G \setminus N$. This contradicts the fact that G/N has a trivial center. ■

Note also that the two topologies cannot be distinguished on the local level of normal closures of finite subgroups: Property (A.2) of the point stabilizer K of a totally imprimitive finitary permutation representation of G shows that $\langle F^G \rangle \cap K$ is open in $\langle F^G \rangle$ with respect to the topology of pointwise convergence.

PROPOSITION 5.3. *Let G be a subgroup of the totally imprimitive subgroup W of $\text{FSym}(\Omega)$.*

(a) *G is dense in W with respect to the topology of pointwise convergence if and only if $G' = W'$.*

(b) *If G is dense in W with respect to the topology of pointwise convergence, and if W has uncountably many totally imprimitive permutation representations, then G too has uncountably many totally imprimitive permutation representations.*

Proof. (a) Consider elements $x, y \in W$. Let $\Gamma = \text{supp}_\Omega x$. Since G is dense in W , there exist $a \in G$ and $b \in W_\Gamma$ such that $y = ab$. It follows that $[x, y] = [x, a][x, b]^a = [x, a]$. Let $\Delta = \text{supp}_\Omega a$. Again there exist $c \in G$ and $d \in W_\Delta$ such that $x = dc$. Hence $[x, y] = [x, a] = [d, a]^c [c, a] = [c, a] \in G'$. This shows that $W' = G'$.

(b) Consider two totally imprimitive permutation representations $\sigma, \tau: W \rightarrow \text{FSym}(\omega)$. It suffices to show that $\sigma|_G = \tau|_G$ implies $\sigma = \tau$. From the above, σ and τ coincide on W' . Consider some $w \in W$. For every $z \in W'$ we have $[w\sigma, z\sigma] = [w, z]\sigma = [w, z]\tau = [w\tau, z\tau] = [w\tau, z\sigma]$. Hence $W'\sigma$ is centralized by $w\sigma \cdot (w\tau)^{-1}$. But the transitive subgroup $W'\sigma$ of $\text{FSym}(\omega)$ has a trivial centralizer in $\text{FSym}(\omega)$. It follows that $w\sigma = w\tau$. ■

Example 4.4 shows that, in the situation of Proposition 5.3, is it impossible to replace the word “uncountably” with the word “countably.”

6. FINITARY IMAGES OF TOTALLY IMPRIMITIVE GROUPS

THEOREM 6.1. *Let G be a totally imprimitive subgroup of $\text{FSym}(\Omega)$, and suppose that M is the kernel of a totally imprimitive finitary permutation rep-*

resentation of G . For every finite subgroup F of G there exists a finite G -block Δ in Ω such that $\text{supp}_\Omega F \subseteq \Delta$ and $\langle F^G \rangle \cap M = \bigcap_{g \in G} (\langle F^G \rangle \cap M) G_{\Delta g}$.

Proof. Let K be the point stabilizer of the given totally imprimitive action of G with kernel M . Consider a fixed finite subgroup F of G . From (A.1) and (A.2') there is a finite G -block Δ in Ω containing $\text{supp}_\Omega F$ such that $K \leq G_{\{\Delta\}}$ and $\langle F^G \rangle \cap G_\Delta \leq K$. Let $\{\Delta_i\}_{i \in \omega}$ denote a tower of finite G -blocks containing Δ , with union Ω . For each i , let

$$(\langle F^G \rangle \cap M)_i = \bigcap_{g \in G} ((F^G) \cap M) G_{\Delta_i g}.$$

Then

$$\bigcap_{i \in \omega} ((F^G) \cap G_\Delta) (\langle F^G \rangle \cap M)_i \subseteq \bigcap_{g \in G} \bigcap_{i \in \omega} ((F^G) \cap G_\Delta) (\langle F^G \rangle \cap M) G_{\Delta_i g}.$$

The right side of the above formula is the closure of $((F^G) \cap G_\Delta) (\langle F^G \rangle \cap M)$ in G with respect to the topology of pointwise convergence. But this product is closed in G as the intersection of the closed subgroup $\langle F^G \rangle$ with the open subgroup $G_\Delta \cdot (\langle F^G \rangle \cap M)$. Hence

$$\bigcap_{i \in \omega} ((F^G) \cap G_\Delta) (\langle F^G \rangle \cap M)_i \subseteq ((F^G) \cap G_\Delta) (\langle F^G \rangle \cap M).$$

However, $\langle F^G \rangle \cap G_\Delta$ has a finite index in $\langle F^G \rangle$, whence

$$((F^G) \cap G_\Delta) (\langle F^G \rangle \cap M)_i = (\langle F^G \rangle \cap G_\Delta) (\langle F^G \rangle \cap M) \quad \text{for some } i.$$

It follows that

$$\begin{aligned} \langle F^G \rangle \cap M &\subseteq ((F^G) \cap M)_i \subseteq \bigcap_{g \in G} ((F^G) \cap G_\Delta)^g ((F^G) \cap M)_i \\ &= \bigcap_{g \in G} ((F^G) \cap G_\Delta)^g ((F^G) \cap M) \subseteq \bigcap_{g \in G} ((F^G) \cap K)^g \\ &= \langle F^G \rangle \cap M. \end{aligned}$$

It remains to replace Δ with Δ_i . ■

In the situation of Theorem 6.1 the condition $\bigcap_{g \in G} (\langle F^G \rangle \cap M) G_{\Delta g} = \langle F^G \rangle \cap M$ is of course also satisfied for every finite G -block Δ^* containing Δ , since $\langle F^G \rangle \cap M \subseteq \bigcap_{g \in G} (\langle F^G \rangle \cap M) G_{\Delta^* g} \subseteq \bigcap_{g \in G} (\langle F^G \rangle \cap M) G_{\Delta g}$.

Proof of Theorem C. (a) Suppose first that G/M has a faithful finitary permutation representation with infinite orbits. If there are just finitely many orbits, then Theorem A yields that M is the kernel of a totally imprimitive finitary permutation representation of G . In this case Theorem 6.1 applies.

Suppose now that there are infinitely many orbits. Since G/M is countable, we have in fact only countably many orbits $\Gamma_0, \Gamma_1, \dots$. Let M_k denote the kernel of the action of G on $\Gamma_0 \cup \dots \cup \Gamma_k$. Again M_k is the kernel of a totally imprimitive finitary permutation representation of G . Consider a fixed finite subgroup F of G , and choose k such that F acts trivially on Γ_l for all $l \geq k+1$. Then Theorem 6.1 provides a finite G -block Δ such that

$$\begin{aligned} \langle F^G \rangle \cap M &\subseteq \bigcap_{g \in G} (\langle F^G \rangle \cap M) G_{\Delta g} \\ &\subseteq \bigcap_{g \in G} (\langle F^G \rangle \cap M_k) G_{\Delta g} = \langle F^G \rangle \cap M_k = \langle F^G \rangle \cap M. \end{aligned}$$

(b) Suppose that M satisfies the condition in part (b) of Theorem C. Let $\{\Delta_i\}_{i \in \omega}$ be a tower of finite G -blocks with $|\Delta_0| = 1$ and with union Ω , and let $F_i = \{g \in G \mid \text{supp}_\Omega g \subseteq \Delta_i\}$ for each i . By assumption, for each i there exists $j_i \geq i$ such that $\langle F_i^G \rangle \cap M = \langle F_i^G \rangle \cap M_{j_i}$, where $M_j = \bigcap_{g \in G} M G_{\Delta_j g}$. Consider the canonical homomorphism

$$\varphi: G \rightarrow \text{Dr}_{i \in \omega} G / \langle F_i^G \rangle M_{j_{i+1}}.$$

It follows as in [5] (see [15, Theorem 2.5]) that $M = \bigcap_{i \in \omega} \langle F_i^G \rangle M_{j_{i+1}} = \ker \varphi$. It suffices to show now that

$$\langle F_i^G \rangle M_{j_{i+1}} = \bigcap_{g \in G} \langle F_i^G \rangle \cdot M \cdot G_{\Delta_{j_{i+1}} g}$$

for each i ; then the open subgroup $\langle F_i^G \rangle \cdot M \cdot G_{\Delta_{j_{i+1}}}$ is the point stabilizer of a totally imprimitive finitary permutation representation of G with kernel $\langle F_i^G \rangle M_{j_{i+1}}$.

For simplicity of notation, we consider a fixed i and let $j = j_{i+1}$. The inclusion

$$\langle F_i^G \rangle M_j = \langle F_i^G \rangle \cdot \bigcap_{g \in G} M G_{\Delta_j g} \subseteq \bigcap_{g \in G} \langle F_i^G \rangle \cdot M \cdot G_{\Delta_j g}$$

is obvious. Conversely, let $x \in \bigcap_{t \in T} \langle F_i^G \rangle \cdot M \cdot G_{\Delta_j t}$ where T denotes a right transversal of $G_{\{\Delta_j\}}$ in G . For each $t \in T$ there exist $f_t \in \langle F_i^G \rangle$, $m_t \in M$, and $h_t \in G_{\Delta_j t}$ such that $x = f_t m_t h_t$. From Lemmata 3.1 and 3.2 there is a cofinite subset T_0 in T such that $x^{t^{-1}} \in G_{\Delta_j}$ for all $t \in T_0$. We may hence assume that $f_t = 1 = m_t$ for all $t \in T_0$. Since $\text{supp}_\Omega F_i \subseteq \Delta_i \subseteq \Delta_j$, we may further assume that $\text{supp}_\Omega f_t \subseteq \Delta_j t$ for all $t \in T \setminus T_0$. Let $f = \prod_{t \in T \setminus T_0} f_t$. Then $x = f \bar{m}_t \bar{h}_t$ for all $t \in T$, where $\bar{m}_t = m_t^{f \cdot f_t^{-1}} \in M$ and $\bar{h}_t = f^{-1} f_t h_t \in G_{\Delta_j t}$. It follows that $x \in \langle F_i^G \rangle \cdot (\bigcap_{t \in T} M G_{\Delta_j t}) = \langle F_i^G \rangle M_j$. ■

We have not been able to decide whether every kernel M of a finitary permutation representation of G with all orbits infinite satisfies the stronger

condition in part (b) of Theorem C. However, the following result shows that this is at least the case when G/M has no proper abelian normal subgroup.

THEOREM 6.2. *Let M be a proper normal subgroup of the totally imprimitive subgroup G of $\text{FSym}(\Omega)$, such that G/M has no nontrivial abelian normal subgroup. Then M is the kernel of some finitary permutation representation with all orbits infinite if and only if for every finite subgroup F of G there exists a finite G -block Δ in Ω such that $\text{supp}_\Omega F \subseteq \Delta$ and $\langle F^G \rangle \cap M = \langle F^G \rangle \cap \bigcap_{g \in G} MG_{\Delta g}$.*

Proof. In the presence of part (b) of Theorem C, it suffices to show the necessity of the condition. Suppose that G/M is a finitary permutation group with all orbits infinite. Consider a finite subgroup F in G , and choose a finite M -invariant G -block Δ in Ω which contains $\text{supp}_\Omega F$. Let $M_0 = \bigcap_{g \in G} MG_{\Delta g}$. Since every element in M_0 acts on Δ like an element from M , we obtain $[F, M_0] = [F, M] \leq M$. Therefore, $(\langle F^G \rangle \cap M_0)M/M$ is an abelian normal subgroup in G/M which must be trivial by the hypothesis. ■

The situation is even better for iterated wreath products of finite transitive permutation groups.

THEOREM 6.3. *Suppose that the proper normal subgroup M of the totally imprimitive subgroup G of $\text{FSym}(\Omega)$ is contained in the normal closure $\langle E^G \rangle$ of some finite subgroup E of G . Then the following are equivalent.*

- (a) *M is the kernel of a totally imprimitive finitary permutation representation.*
- (b) *M is the kernel of a finitary permutation representation with all orbits infinite.*
- (c) *There exists a finite G -block Δ in Ω such that $M = \bigcap_{g \in G} MG_{\Delta g}$.*
- (d) *M is the normal closure in G of a finite group.*

Proof. If M is the kernel of some finitary permutation representation with all orbits infinite, then part (a) of Theorem C readily gives a finite G -block Δ such that $\text{supp}_\Omega E \subseteq \Delta$ and

$$M = \langle E^G \rangle \cap M = \bigcap_{g \in G} (\langle E^G \rangle \cap M)G_{\Delta g} = \bigcap_{g \in G} MG_{\Delta g}.$$

This equation implies that $M = \langle F^G \rangle$ where $F = \{m \in M \mid \text{supp}_\Omega m \subseteq \Delta\}$. Finally, if $M = \langle F^G \rangle$ for some finite $F \leq G$, we choose a finite G -block Δ containing $\text{supp}_\Omega F$. Then M is contained in the normal subgroup N of all elements in G which stabilize all G -translates of Δ setwise. We consider G as a subgroup of $X \text{ wr}_\Sigma \text{FSym}(\Sigma)$ where Σ is the set of N -orbits in Ω ,

and where X is the subgroup of $\text{Sym}(\Delta)$ induced from the action of $G_{\{\Delta\}}$. Then G/M is isomorphic to a subgroup of the totally imprimitive finitary permutation group $X/\langle F^X \rangle \text{wr}_{\Sigma} \text{FSym}(\Sigma)$. ■

Examples of nonfinitary quotients in perfect totally imprimitive finitary permutation groups will be provided now.

EXAMPLE 6.4. For any prime p , the derived subgroup of the infinite iterated wreath product $G = C_p \text{wr} C_p \text{wr} C_p \text{wr} \cdots$ of cyclic groups of order p contains a normal subgroup M such that the quotient G'/M has a trivial center and no faithful finitary permutation representation.

Proof. Let $G = F_0 \text{wr} F_1 \text{wr} F_2 \text{wr} \cdots$, where each F_i is a copy of the cyclic group of order p . Then G is the union of the ascending chain of subgroups $G_n = F_0 \text{wr} \cdots \text{wr} F_n$ ($n \in \omega$), where G_n is the 1-component of G_{n+1} . Let $B_1 = \zeta(G_1) \leq \langle F_0^G \rangle \cap G'$, let $B_n = \langle B_1^{G_n} \rangle$ for $n \geq 1$, and let $B = \bigcup_{n \geq 1} B_n = \langle B_1^G \rangle = \langle B_1^{G'} \rangle$. We construct the subgroup $A = \bigcup_{n \geq 1} A_n$ of B recursively via $A_1 = 1$ and $A_{n+1} = \langle A_n^{G_{n+1}}, a_{n+1} \rangle$, where the element $a_{n+1} \in B_{n+1} \setminus \langle A_n^{G_{n+1}} \rangle$ is so chosen that $[a_{n+1}, G_{n+1}] \leq \langle A_n^{G_{n+1}} \rangle$. Since each A_n is normal in G_n , the group A is a normal subgroup in G .

Our next aim is it to show that $A_n = B_n \cap A$ for every n . Note that $\langle B_n^{G_{n+1}} \rangle = B_n \times \cdots \times B_n$ is just the direct product of p copies of B_n , which are permuted cyclically by the cycle $c = (12 \cdots p)$ in the top group of $G_{n+1} = G_n \text{wr} C_p$. Therefore we can write $a_{n+1} = (b_{n,1}, \dots, b_{n,p})$ with $b_{n,i} \in B_n$. By construction we have $[a_{n+1}, c] \in \langle A_n^{G_{n+1}} \rangle$, so that $b_{n,i}^{-1} b_{n,i+1} \in A_n$ for all i . Assume there is an index i such that $b_{n,i} \in A_n$. Then $b_{n,i+1} \in A_n$ too, whence $b_{n,i} \in A_n$ for all i , and $a_{n+1} \in \langle A_n^{G_{n+1}} \rangle$. This contradiction shows that $b_{n,i} \in B_n \setminus A_n$ for all i . In particular, $B_n \cap A_{n+1} = B_n \cap \langle A_n^{G_{n+1}} \rangle = A_n$. Thus induction gives

$$B_n \cap A_{n+j+1} = B_n \cap B_{n+j} \cap A_{n+j+1} = B_n \cap A_{n+j} = A_n \quad \text{for all } j \in \omega,$$

and this yields $A_n = B_n \cap A$.

As explained at the beginning of Section 4, the group G has a natural finitary permutation action on a countable set Ω , and the supports Δ_n of the groups G_n form a tower of finite G -blocks with union Ω . We shall show next that $A \neq M_n = \bigcap_{g \in G'} AG'_{\Delta_n g}$ for each n . To this end we consider $a_{n+1} = (b_{n,1}, \dots, b_{n,p})$ as above. Since $b_{n,1} \in B_n \setminus A_n$ it is obvious that $(b_{n,1}, 1, \dots, 1) \in M_n \setminus A$. In fact, we even have $(b_{n,1}, 1, \dots, 1) \in M_n \setminus M_{n+1}$ for every n , so that $\{M_n\}_{n \in \omega}$ is a strictly descending chain. It is evident now that the subgroup $M = \bigcap_{n \in \omega} M_n$ of $\langle B_1^{G'} \rangle$ is a normal subgroup in G' which fails to satisfy the condition in Theorem 6.3. By Proposition 5.2, the quotient G'/M has a trivial center. ■

Example 6.4 is based on the fact that every finite p -group has a nontrivial center.

QUESTION. *Do there also exist nonfinitary quotients in perfect totally imprimitive finitary permutation groups which are not a p -group for some prime p ?*

We shall finally give necessary conditions for the structure of finitary linear quotients of totally imprimitive finitary permutation groups G . Note that every periodic abelian group is finitary linear, since its divisible hull embeds into the multiplicative group of a direct power of copies of the algebraic closure of the field \mathbb{Q} . We shall therefore concentrate on quotients modulo intransitive normal subgroups of G .

THEOREM 6.5. *Let M be an intransitive normal subgroup of the totally imprimitive finitary permutation group G . If G/M is an irreducible group of finitary transformations, then G/M is a totally imprimitive finitary permutation group.*

Proof. Let \tilde{M} denote the group of all elements in G which leave the M -orbits invariant. The quotient G/\tilde{M} of G/M is a totally imprimitive finitary permutation group on an infinite set. Therefore [2, (4.2) and (4.3)] and [7, Theorem B] imply that the irreducible finitary linear group G/M is nonlinear and imprimitive. In particular there exist a finite-dimensional vector space V_0 and an infinite set Σ such that G/M can be considered as a subgroup of the wreath product $W = L \text{ wr}_{\Sigma} \text{FSym}(\Sigma)$, where $L \leq \text{GL}(V_0)$ is an irreducible linear group, and where W acts canonically on $V = V_0 \otimes \Sigma$. Moreover, G/M projects onto a transitive subgroup of the top group $\text{FSym}(\Sigma)$. The normal subgroup $\{g \in G \mid gM \in L^{(\Sigma)}\}$ of G does not contain G' ; therefore it is intransitive and of finite exponent. A well-known result of Burnside (see [16, 1.23]) yields that L is finite. In particular, W is a finitary permutation group in its natural action on $L \times \Sigma$. But G/M acts transitively on Σ . So $L \times \Sigma$ has just finitely many (G/M) -orbits, and they are infinite. Now Theorem A gives the assertion. ■

Since every finitary linear group with a trivial unipotent radical is a subdirect product of irreducible finitary linear groups [12, Proposition 1], it follows immediately that every unipotent-free finitary linear quotient of the totally imprimitive finitary permutation group G is a finitary permutation group.

COROLLARY 6.6. *Let M be an intransitive normal subgroup of the totally imprimitive finitary permutation group G . If G/M is a group of finitary transformations, then G/M is the extension of a nilpotent normal p -subgroup of finite exponent by a finitary permutation group.*

Proof. Define \tilde{M} as in the proof of Theorem 6.5. Consider the intersection N/M of \tilde{M}/M with the unipotent radical of the finitary linear group G/M . Then G/N is a finitary permutation group. If the characteristic p of the underlying field is zero, the section N/M is trivial. Otherwise it is a p -group. The intransitive normal subgroup N of G is a subdirect power of a fixed finite group F . Therefore the nilpotency class of every finite subgroup of the p -section N/M is bounded by a function of F . In particular N/M is nilpotent and of finite exponent. ■

In the above situation, there exists a (G/M) -series of finite length and with elementary-abelian factors in N/M . And since G is a finitary permutation group, the conjugation action of G/M on each of these factors is finitary linear. It remains open whether every quotient of G with these properties is finitary linear.

ACKNOWLEDGMENTS

Each of the two authors thank the university of his coauthor for inviting him to a visit, during which essential parts of the work on this paper could be carried out.

REFERENCES

1. V. V. Belyaev, Finitary linear representations of infinite symmetric and alternating groups, *Algebra and Logic* **32** (1993), 319–327.
2. B. Bruno and R. E. Phillips, Residual properties of finitary linear groups, *J. Algebra* **166** (1994), 379–392.
3. P. J. Cameron, Metric and topological aspects of the symmetric group of countable degree, *European J. Combin.* **17** (1996), 135–142.
4. J. I. Hall, Infinite alternating groups as finitary linear transformation groups, *J. Algebra* **119** (1988), 337–359.
5. P. Hall, Periodic FC -groups, *J. London Math. Soc.* **34** (1959), 289–304.
6. P. Hall, Wreath powers and characteristically simple groups, *Proc. Cambridge Philos. Soc.* **58** (1962), 170–184.
7. F. Leinen and O. Puglisi, Countable recognizability of primitive periodic finitary linear groups, *Math. Proc. Cambridge Philos. Soc.* **121** (1997), 425–435.
8. J. D. P. Meldrum, “Wreath Products of Groups and Semigroups,” Longman Group, Essex, England, 1995.
9. P. M. Neumann, The lawlessness of groups of finitary permutations, *Arch. Math.* **26** (1975), 561–566.
10. P. M. Neumann, The structure of finitary permutation groups, *Arch. Math.* **27** (1976), 3–17.
11. R. E. Phillips, The structure of groups of finitary transformations, *J. Algebra* **119** (1988), 400–448.
12. R. E. Phillips, Finitary linear groups: a survey, in “Finite and locally finite groups,” (B. Hartley, G. M. Seitz, A. V. Borovik, and R. M. Bryant, Eds.), pp. 111–146. NATO ASI Series C **471**, Kluwer Academic, Dordrecht/Boston/London, 1995.

13. R. E. Phillips and J. G. Rainbolt, Images of periodic linear groups, *Arch. Math.* **71** (1998), 97–106.
14. D. Segal, Normal subgroups of finitary permutation groups, *Math. Z.* **140** (1974), 81–85.
15. M. J. Tomkinson, “FC-groups,” Pitman, Boston/London/Melbourne, 1984.
16. B. A. F. Wehrfritz, “Infinite Linear Groups,” Springer Verlag, Berlin/Heidelberg/New York, 1973.
17. B. A. F. Wehrfritz, Faithful linear representations of certain free nilpotent groups, *Glasgow Math. J.* **37** (1995), 33–36.
18. B. A. F. Wehrfritz, Periodic normal subgroups of linear groups, *Arch. Math.* **71** (1998), 169–172.
19. B. A. F. Wehrfritz, Finitary linear images of finitary linear groups, *Mathematica Pannonica*. **10**, No. 2 (1999), 277–290.
20. J. Wiegold, Groups of finitary permutations, *Arch. Math.* **25** (1974), 466–469.
21. H. Wielandt, “Unendliche Permutationsgruppen,” Vorlesungen an der Universität Tübingen, 1959/60.¹

¹See also H. Wielandt, “Mathematische Werke /Mathematical Works,” Vol. I (B. Huppert and H. Schneider, Eds.), de Gruyter, Berlin/New York, 1994.